Prof. Enrique Mateus-Nieves
PhD en Educación Matemática
DOI: 10.13140 / RG.2.2.15784.90886

Four weaknesses found in higher education students learning improper integrals

The concept of improper integral is important for students who are professionally trained in science (mathematics, biology, chemistry) or engineering due to the number of applications it has, among them: calculation of probabilities to define functional rules, calculation of Fourier and Laplace transforms; physical calculations (work, energy... in certain circumstances). To name but a few.

In particular, improper integrals become interesting when they are used to analyse the character of a series of positive terms through the integral criterion and where, from the solution of the improper integral associated to the series, the character of the series can be determined and vice versa. One of the needs revealed in the work of the laboratory that I direct, lies in the various manifestations of university professors, related to the lack of clarity that students have about the application of content involving integrals, particularly improper integrals. They show a lack of connection with the environment and with everyday life problems. In addition, they state that there is little literature on the subject in Mathematics Education.

Aware of this reality, this research laboratory has been carrying out different works related to the teaching and learning processes of infinitesimal calculus, in particular integral calculus. Among them: Mateus-Nieves (2011, 2015, 2016 and 2019*a*, 2019*b*) shows integral calculus, framed within infinitesimal calculus.

Mateus-Nieves (2016) carries out a didactic analysis of how three groups of university students learn to use the method of integration by parts (MIP). She identifies difficulties in the students in recognising the type of integral involved in the situation posed. It shows mechanical use of the IPM algorithm, through repetitive exercises that allow us to infer little understanding of it. It is emphasised that it is possible for students to understand and apply it solidly if they identify the relationship between the type of integral, the method to be applied and the relational structure of the problems to which it is applied.

Mateus-Nieves (2019*b*) presents an extension to the 2016 research, with a study of the integral from three dimensions: epistemic, to address the historical genesis of the concept. Cognitive, to consider the difficulties that students present when faced with problem situations involving the concept. And didactic, where he proposes problem situations that aim to enrich the field of social practices that are addressed with the concept. From this study, it indicates that the mathematical notions for the integral have a high level of abstraction, fundamental in the development of advanced mathematics and difficult for students to learn, therefore, a study that takes into

Citation: Mateus-Nieves, E. (2021). Four weaknesses found in higher education students learning improper integrals. *Lecture: Mathematics Education, Project: PhD research line in Mathematics Education*. Bogotá. D. C.
DOI: 10.13140 / RG.2.2.15784.90886.

account the logical conditions involved in the process of constitution of this mathematical object, contributes to a better understanding of it and its possible articulation.

Mateus-Nieves and Hernández (2020) present a study conducted with three groups of university students learning Integral Calculus. They redesign the curricular structure of the Calculus II course, articulating the epistemic, cognitive and didactic dimensions for various meanings identified for the integral, seeking those students make sense of what they call "partial meanings for the integral" (p. 205). Enabling them to show the integral as a problem-solving tool, where students reach a level of abstraction, fundamental in the development of advanced mathematics. They identify the implicit perception of a culture in the teacher and in the student, where learning to say what the integral is and to represent it geometrically, without having an understanding of it, is enough to pass the subject; presence of a formal-mechanistic approach.

Mateus-Nieves and Rojas (2020), from a case study, show that the teaching of Infinitesimal Calculus begins in secondary education and is formalised in the first semesters of university, where it is expected to move from elementary mathematical thinking to advanced mathematical thinking. Given that in higher education the progressive theatricalization implies the need to abstract, define, analyse and formalise; among the cognitive processes of psychological component, in addition to abstracting, stand out those of "representing, conceptualising, inducing and visualising" (p. 69). In this way, the aim is for students to "understand" this type of concepts through the exercise of algorithms specific to the subject and its properties.

Mateus-Nieves (2020) presents a reflection on working with improper integrals where he highlights that traditional teaching does not lead students to acquire the ability to understand that there is a mathematical object called "integral", which in turn is unitary and systemic. That it is made up of various meanings (types of integral), and that these can be used in various situations of both an intra and extra mathematical nature. It indicates that traditional teaching (mechanistic approach), evaluates students in such a way that they only apply an algorithm, iteratively, without understanding what they are doing or the benefits of using this tool. This type of teaching suggests that the important thing is to master the procedures for solving exercises, or to memorise definitions and understand the proof of theorems (formal approach), leaving aside the usefulness and richness of this mathematical entity for problem solving.

With the teaching and learning processes of university mathematics, it is expected that at the end of courses such as: Calculus I, II, III and Mathematical Analysis, students will have acquired skills in the handling and use of concepts, theorems and fundamental procedures of integral calculus that will enable them to become competent in particular situations in the working environment that involve the use of different types of integral. In the aforementioned works, it has been identified that, in educational practice, it is observed that improper integrals are approached from oscillating integrals, i.e., on situations adjusted so that the function is integrable; very few situations are approached that can be related to the everyday life of a university student, leaving the task to the student to solve the numerical problems that these problems bring with them. It was identified that the students do not manage to adequately understand these concepts and relate them to some previously studied knowledge (such as sequences, series and definite and indefinite integrals), because these concepts, worked on in their first year of university, have been poorly grounded. First weakness.

We found that the formal-mechanistic approach to teaching infinitesimal calculus persists, where university teaching traditionally focuses on the student memorising criteria and working on the resolution of multiple exercises involving integrals, in our case today, improper integrals; However, the interpretation given to the results obtained are minimal, the paradoxes that may appear are not considered, nor are similar or analogous counterexamples sought that allow the student to go deeper into the concepts, deepen their knowledge and investigate in analogous situations that allow them to reach an effective level of mathematical competence for their professional performance. Second weakness.

Under this model, we observe that several students learn to use the integral as a tool, use some concepts related to improper integration but decontextualised and disconnected from contents learnt in previous courses. They do not perceive any connection or articulation between them, nor do they see any relationship. We find that they limit themselves to memorising a set of criteria and techniques that, if they were contextualised, would have greater meaning for them. We believe that part of the responsibility for this situation lies in inadequate teaching practices (third weakness), given that we noticed that they focus their interest in teaching these subjects according to what the textbooks indicate, they are taken as a road map and are limited to the few situations presented there, leaving some of them decontextualised, where the use of mathematical support software is limited (fourth weakness) and in the cases where they do use it, its use is limited to geometric modelling, introduced as didactic work.

The progress achieved so far in this research shows that analysing the first examples of improper integrals, studying the Cauchy condition for convergence of series and integrals, applying the integral criterion for convergence of series is a challenge for students, particularly engineering students, who end up mechanically repeating algorithms and processes that lead to a solution that rarely validates whether and to what degree it responds to the request of the proposed problem situation. We are currently preparing an article with these results that we will soon share. For example: applying the first and second comparison criteria, both for improper integrals of the first and second kind. It is difficult for students to determine how they interact dynamically with each other, so that any algebraic expression introduced can be represented graphically from the different tools available to the teacher to explain the subject.

On the other hand, it is overlooked that situations where mathematical software is used make it easier for the student to identify this relationship, a situation that is rarely applied in practice. Perhaps due to a lack of resources, perhaps due to a lack of time to cover the different topics in the syllabus of the subject. To some extent, once the appropriate comparison function has been chosen, it is possible to have an indication of the character of the integral. But in many cases a numerical estimate of that value is necessary. For this purpose, an approximate calculation option is available (generated by the programme, although an internal numerical method must be used) so that at each step a Riemann integral is calculated, generating a finite sequence of values that present successive numerical approximations. The corresponding numerical option for each improper integral is available only if the option exact integral. Modifying the comparison function could lead to the situation that this limit is a non-zero number, in which case the horizontal asymptote line of the graph of the quotient function appears and the message that both improper integrals have the same character. This situation is not very understandable for students who cannot visualise the fact. Therefore, we recommend teachers to present a list of guided experiences with some integrating functions so that students become familiar with the modelling environment and subsequent mathematical simulation, experimental and that allows them to achieve meaning of the mathematical object they learn, that is development of mathematical competence to apply this type of integrals to both intra and extra mathematical context proper to their professional work.

Observations: We have identified from comments of students participating in this ongoing research, that the elements offered from this laboratory have facilitated their understanding of the integral as a mathematical entity that is complex by nature. This leads us to infer the difficulty of learning to use this type of integral, especially when the integrand function does not have a known primitive function. As we mentioned at the beginning of today's lecture. We ratify what was stated by the university professors who are with us in relation to the scarce existing literature, published research works focused on the processes of teaching and learning the concept of the integral. Although the integral appears implicitly in some works, it is generally developed within the framework of the definite integral, but improper integrals are not presented as an extension of definite integrals. What is observed is the teaching of isolated entities, disjointed from each other, generating an absence of meaning for the integral in the students.

In this research, we have observed that both teachers and students tend to substitute the extreme values of the integral and do not calculate the integrals using limit processes, but rather generalising Barrow's rule and directly substituting the extremes of the integral in the variable, even when one of the extremes is infinite. These reasons led us to limit our initial field of investigation and to opt only for the use of the graphic and algebraic registers of representation in a first approach to the analysis of the students' understanding of some basic concepts of improper integration, under the hypothesis that the students are only used to working in one of them (the algebraic). In basic university mathematics courses, the challenge is to incorporate problem situations that reflect the use of concepts and calculations to interpret the solutions, which in turn favours the development of fundamental skills, knowledge and values.

We seek to identify difficulties that students encounter when learning concepts related to improper integration; it has been found that some of them seem inherent to the concept of improper integral itself and others are related to the absence of meaning or to other concepts inherent to integral calculus, such as function, limit, antiderivative, among others, which have already been recorded in the literature in Mathematics Education. From here we consider that mathematical modelling is presented as a didactic strategy that allows both teachers and students to simulate and interpret different problems and situations of real or academic life, highlighting different conditions of application of the contents of university mathematics courses. This strategy seeks to solve the need to show and manipulate applications of the contents of improper integrals, Taylor polynomials, polar coordinates and conic sections in problems with concrete examples in the areas of biological sciences and engineering. The invitation of the members of this laboratory and the university professors who are with us today is to continue with this work in order to offer didactic elements and strategies to professors interested in enriching their university teaching practice, a subject that is very little addressed.

Bibliographical references

Mateus-Nieves, E. (2011). Epistemology of the derivative as the foundation of differential calculus. *Voces y Silencios Revista Latinoamericana de Educación*, 2 (especial):3-2. DOI:10.18175/vys2. especial. 2011.01. Bogotá. Colombia.

Mateus-Nieves, E. (2015). Evolución histórico-epistemológica del concepto de integral [Historical-epistemological evolution of the integral concept]. Project: line of research PhD in Mathematics Education. DOI:10.13140/RG.2.2.32081.56164.

Mateus-Nieves, E. (2016). Análisis Didáctico a un Proceso de Instrucción del Método de Integración por Partes, [Didactic Analysis of an Instructional Process of the Integration by Parts Method]. ISSN: 1980-4415-bolema-30-55-0559. BolemaBoletimdeEducaçãoMatemática30:559-585. DOI:10.1590/1980-4415v30n55a13.

Mateus-Nieves, E. (2019*a*). Extensiones al concepto de integral. [Extensions to the concept of integral]. Conference: Didactics of calculus (infinitesimal). Bogotá-Colombia. DOI:10.13140/RG.2.2.34282.95689. Project: line of research PhD in Mathematics Education.

Mateus-Nieves, E. (2019*b*). Breve reseña de la integral. [Brief overview of the integral]. Conference: Didactics of calculus (infinitesimal).Bogotá-Colombia. Nov.2019. *Project: line of research PhD in Mathematics Education*. DOI: 10.13140/RG.2.2.19373.51683.

Mateus-Nieves, E., & Rojas Jiménez, C. (2020). Mathematical generalization from the articulation of advanced mathematical thinking and knot theory. Acta Scientiae. (Canoas),22(3), 65-81, Maio/Jun. 2020. ISSN: 2178-7727. DOI: 10.17648/acta.scientiae.5667.

Mateus-Nieves, E., & Hernández Montañez W. (2020). Significado Global de la Integral Articulando su Complejidad Epistémica. [Global Meaning of the Integral Articulating its Epistemic Complexity]. UNION, *Revista Iberoamericana de Educación Matemática*. ISSN 1815-0640. Año XVI (60), pp. 196-211.

Mateus-Nieves, E. (2020). Preliminares del cálculo integral. [Preliminaries of integral calculus]. Bogotá-Colombia. Agos.2019. Project: line of research PhD in Mathematics Education. DOI: 10.13140/RG.2.2.19373.51683