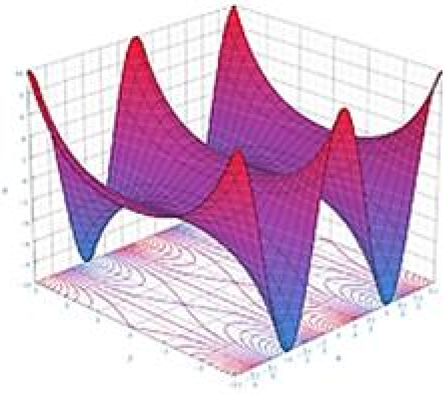
Maclaurin series (Taylor around 0) notable



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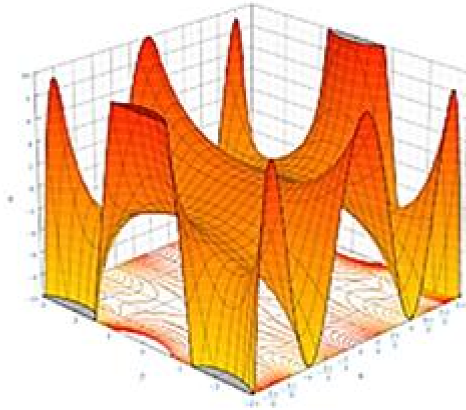
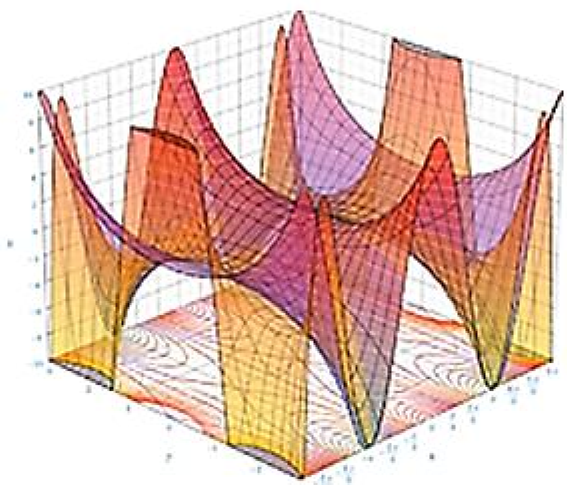
The cos x function



one eighth order approximation of the cosine function in the complex plane



The two previous images put together

Below are some of Taylor series of basic functions. All developments are also valid for complex values of x.

Exponential function and natural logarithm

$$e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}, \quad \forall\, x; n \in IN_0$$

$$ln\ (1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \ \ for \quad |x| < 1$$

Geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1$$

Binomial theorem

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(n-\alpha)} x^n$$ For $|x| < 1$ and any α complex

Trigonometric functions

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad , \forall x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad , \forall x$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} \quad , \ \textbf{\textit{For}} \quad |x| < \frac{\pi}{2}$$

Where Bs are the Bernoulli numbers.

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} \quad , \text{ para } |x| < \frac{\pi}{2}$$

$$\csc x = \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)B_n x^{2n-1}}{(2n)!} \quad , \textit{ For } \; 0<|x|<\pi$$

$$\arcsin x = \sum^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \quad , \textit{ For } \; |x|<1$$

$$\arctan x = \sum^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad , \textit{ For } \; |x|<1$$

hyperbolic functions

$$\sinh x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} \quad , \forall x$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} \quad , \forall x$$

$$\tanh x = \sum_{n=1}^{\infty} \frac{B_{2n} 4^n (4^n - 1)}{(2n)!} x^{2n-1} \quad , \textit{ For } \; |x| < \frac{\pi}{2}$$

$$\sinh^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \quad , \textit{ For } \; |x|<1$$

$$\tanh^{-1} x = \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1} \quad , \textit{ For } \; |x|<1$$

W de Lambert functions

$$W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n \quad , \textit{ For } \; |x| < \frac{1}{e}$$

Los E_k del desarrollo de *sec(x)* son Números de Euler. The numbers appearing in Bk developments tan (x) and tanh (x) are Bernoulli numbers. The values C (α, n) are binomial expansion of the binomial coefficients. The Ek developments of sec (x) are Euler numbers.

Several variables

The Taylor series can be generalized to functions of variables:

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_d}}{\partial x_d^{n_d}} \frac{f(a_1, \cdots, a_d)}{n_1! \cdots n_d!} (x_1 - a_1)^{n_1} \cdots (x_d - a_d)^{n_d} =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{n_1+\cdots+n_d=n} \binom{n}{n_1 \cdots n_d} \frac{\partial^n f(a_1, \cdots, a_d)}{\partial x_1^{n_1} \cdots \partial x_d^{n_d}} (x_1 - a_1)^{n_1} \cdots (x_d - a_d)^{n_d},$$

Where $\binom{n}{n_1 \cdots n_d}$ is a multinomial coefficient. As an example, for a function of two variables, x and y, the Taylor series of the second order in a neighborhood of the point (a, b) is:

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$+\frac{1}{2}\left(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right).$$

A Taylor polynomial of second degree can be compactly written as:

$$T(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x}-\mathbf{a})^T \nabla f(\mathbf{a}) + \frac{1}{2}(\mathbf{x}-\mathbf{a})^T \nabla^2 f(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \cdots$$

Where $\nabla f(\mathbf{a})$ is the gradient and $\nabla^2 f(\mathbf{a})$ the Hessian matrix. Otherwise:

$$T(\mathbf{x}) = \sum_{|\alpha| \geq 0} \frac{\mathrm{D}^\alpha f(\mathbf{a})}{\alpha!} (\mathbf{x}-\mathbf{a})^\alpha$$

Applications

Besides the obvious application of polynomial functions used instead of more complex functions to analyze the local behavior of a function, the Taylor series have many other uses. Some of them are: boundary analysis and parametric studies of the same, irrational numbers delimiting estimate its error, L'Hopital theorem for solving indeterminate boundaries, study of stationary points in functions

Some of them are: boundary analysis and parametric studies of the same, irrational numbers delimiting estimate its error, L'Hopital theorem for solving indeterminate boundaries, studying acting stationary points (maximum or minimum points on chairs or strictly increasing or decreasing trend), estimation of integrals, determining convergence and sum of some important series, study and main parameter order infinitesimals, etc..

Anyone wishing to pursue the subject can also check: Math Series, Laurent Series

Bibliography.

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2. Boyer, C. and Merzbach, U. (1991) *A History of Mathematics*. John Wiley and Sons. pp. 202-203.
3. «Neither Newton nor Leibniz - The Pre-History of Calculus and Celestial Mechanics in Medieval Kerala». *MAT 314*. Canisius College.